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Proceedings of the American Academy of Arts and Sciences.

Vol. 68. No. 10.—September, 1933.

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Summary.

The type of circuit considered is that containing a resonating element of very low damping constant, consisting of resistance, inductance, and capacitance in series, in parallel with which is a second capacitance. Although the method was developed with particular reference to the piezo-electric resonator, it is equally applicable to any type of alternating current network containing a sharply tuned resonating element.

The distinguishing feature of the present method lies in the use of a single circular locus for both admittances and impedances. The process of performing an inversion, instead of requiring the construction of a new diagram, now demands nothing more than a change in the value of the unit vector. The solution of problems is thus greatly simplified, and the effect of varying the frequency or any other parameter is easily followed.

The frequency-calibration of the circular locus is described, and the properties of the quadrantal points and especially of the points of maximum and minimum admittance are discussed. The effects upon the performance of the circuit of the addition of fixed impedances in series or parallel with the resonating element are considered. As illustrations of the method the theory of the determination of the constants of the resonating element from observations of the maximum and minimum impedance of the entire circuit is given, and also the problem of finding the length of air-gap of a piezo resonator for maximum voltage across the gap.

Symbols.

$R\ L\ C\ C_1$	Constants of resonating element, Fig. 1.
Z_2	Impedance in series with resonating element, having components R_2 and X_2. Z_2 may be a pure capacitance C_2.

C_3 Parallel capacitance, with reactance X_3 and susceptance b_3

$Z\ Y\ X\ g\ b$ Impedance, admittance, reactance, conductance, and susceptance of the RLC branch.

Z_1' and components R_1', X_1' Impedance of $RLCC_1$ mesh.

C_1' Equivalent series capacitance of $RLCC_1$.

Y_1' and components g_1', b_1' Admittance of $RLCC_1$ mesh.

$R_1''\ X_1''\ C_1''$ Equivalent parallel parameters of resonating element $RLCC_1$.

$R'\ L'\ C'\ C_1'$ Constants of network equivalent to $RLCC_1C_2$.

$Z_m\ Z_n\ Y_m\ Y_n$ Maximum and minimum values of impedance and admittance of $RLCC_1$

Z_2' and components R_2' and X_2' Resultant values for $RLCC_1Z_2$ network.

Y_2' and components g_2', b_2' Resultant values for $RLCC_1Z_2$ network.

$R_2''\ X_2''\ C_2''$ Equivalent parallel values for $RLCC_1Z_2$ network.

ρ Radius of circle.

$a\ a'\ a''$ Distance from point A on diagrams to origin of vectors F or F'.

s Scale-value for impedance diagram for RLC branch.

s_y Scale-value for admittances, RLC or $RLCC_1$ mesh, also with C_3 in parallel.

s_z Scale-value for impedances, $RLCC_1$ mesh, also with Z_2 or C_2 in series.

s_y' Scale-value for admittances, $RLCC_1C_2$ network.

s_y'' Scale-value for admittances, $RLCC_1Z_2$ network.

σ Scale-value for frequencies.

$f_o\ \omega_o$ Frequency and angular velocity corresponding to maximum admittance of RLC branch.

$f_{ym}\ f_{yn}$ Frequencies corresponding to Y_m and Y_n respectively.

$f_{q1}\ f_{q2}$ Frequencies at upper and lower quadrantal points respectively on admittance circle.

$2n_q$ Difference between the frequencies f_{q2} and f_{q1}.

α Damping constant $= R/2L$.

Other special symbols are defined in the text.

Introduction.

The method of geometrical inversion as applied to circular loci in alternating current problems is well known.[1] The particular development about to be described was occasioned by the study of the piezo-electric resonator, but it is of sufficient generality to be applicable to any network containing a resonating element that possesses a damping constant small in comparison with that of any other portion of the circuit. It offers the advantage of rapid and simple solution of problems. The method is also applicable to mechanical or acoustical resonating systems.

In the past it has been customary, when performing inversions of vectors or of their loci, to construct a separate diagram for each inversion. The advantage in the present method lies in carrying out all operations on a single diagram. This involves a departure from convention, in that all vectors representing inductive quantities are drawn upward (in the positive direction), and capacitive downward, whether the quantities are impedances or admittances. This feature is, however, in conformity with the procedure commonly observed in the theory of geometrical inversion. The use of the same circular locus for both impedances and admittances requires also that the unit vector, or scale-value, be introduced explicitly in each equation. Problems involving damped free oscillations and products of complex vectors are not considered.

Development of the Method.

1. We are concerned chiefly with the type of circuit shown in Fig. 1. We proceed to find a general graphical solution for the resultant impedance and admittance, with their components, and also for the selectivity of the network, as functions of the frequency. The *RLC* branch, with its parallel capacitance C_1, is the resonating element.

[1] For the mathematical principles of inversion see, for example, Graustein, Introduction to Higher Algebra, N. Y. 1930, or Ziwet and Hopkins, Analytic Geometry and Principles of Higher Algebra, N. Y. 1922. For applications to alternating currents, see Lee, Graphical Analysis of Alternating Current Circuits, Baltimore 1928. Applications to vibrating systems: Kennelly, Electrical Vibration Instruments, N. Y. 1923. Applications to piezo-electric resonator circuits: Vigoureux, Quartz Resonators and Oscillators, London 1931; D. W. Dye, Proc. Phys. Soc. Lond. 38, pp. 399–457, 1926; Y. Watanabe, Elek. Nachr. Techn. 5, pp. 45–64, 1926, or Proc. Inst. Radio Eng. 18, pp. 695–717 and 862–893, 1930.

In the case of the piezo resonator, R, L, C and C_1 are the equivalent electrical constants introduced by Van Dyke and Dye.[2] Z_2 may represent the air-gap between crystal and electrodes, and C_3 the capacitance of mounting and leads, or, as proved below, $RLCC_1$ may include these capacitances, while Z_2 and C_3 represent the constants of other parts of the circuit.

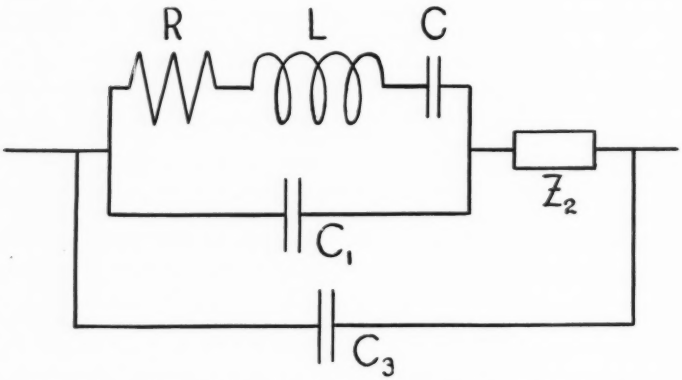

FIGURE 1. Electrical Network Including the Resonating Element $RLCC_1$.

2. *The Resonating Element.* This is the $RLCC_1$ portion of Fig. 1. As starting point we take the impedance of the RLC branch. In Fig. 2, let the resistance vector be represented by any convenient horizontal distance AB. The unit vector, or scale-value, is then $R/\mathrm{AB} = s$ (ohms per division). By "division" is meant unit distance on the diagram. We now write

$$\begin{aligned} R &= s\cdot\mathrm{AB} \\ X &= \omega L - \frac{1}{\omega C} = s\cdot\mathrm{BS} \\ Z &= s\cdot\mathrm{AS} \end{aligned} \tag{1}$$

[2] K. S. Van Dyke, Phys. Rev. 25, p. 895, 1925 and Proc. Inst. Radio Eng. 16, pp. 742–764, 1928; D. W. Dye, Proc. Phys. Soc. Lond. 38, pp. 399–457, 1926.

As the frequency varies, the locus of the impedance vector is of course the line BS. B is the point of resonant frequency f_o, at which the reactance $X = 0$, that is $\omega_o^2 LC - 1 = 0$.

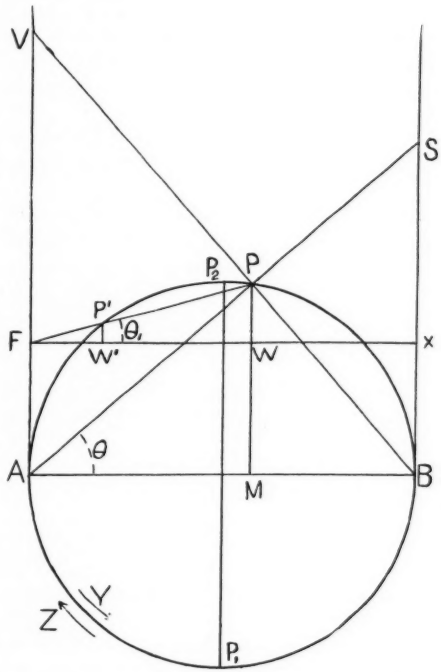

FIGURE 2. Circle Diagram, showing Admittance and Impedance of Resonating Element.

The circular locus of the admittance Y for RLC is found by inverting AS with respect to the point A as center. According to the usual practice, we should assign the value unity to the radius of the circle of inversion, write $Z = AS$, $Y = 1/AS$, and plot a circular locus for Y to some convenient scale, the arguments of the vectors having signs opposite to those for Z. Instead of this, we take AB as radius of the

circle of inversion, with center at A. It is unnecessary to draw this circle in the figure, for it is evident from the principles of inversion theory that under these circumstances the straight line BS inverts into a circle having AB as diameter. The inverse of any point S is P, where AS cuts the circle, while the point B inverts into itself. Hence with suitable choice of scale-value we may use the circle APB as the admittance locus for *RCL*. AP now represents the admittance $Y = 1/Z$, by the equation $Y = s_y \cdot \mathrm{AP}$. Here s_y is the admittance scale-value in ohms per division. Since $\mathrm{AP} \cdot \mathrm{AS} = 1/(\mathrm{AB})^2$, it follows that $ss_y = 1/(\mathrm{AB})^2 = 1/4\rho^2$, where ρ is the radius of the circle APB. Thus s_y is determined in terms of s and ρ. Or, in terms of R and ρ, we have by (1), $s_y = 1/2\rho R$. Hence

$$\frac{1}{R} = s_y \cdot 2\rho \tag{2}$$

For the admittance, conductance and susceptance of *RLC* at any frequency f we have, according to Fig. 2,

$$Y = \frac{1}{Z} = s_y \cdot \mathrm{AP}$$

$$g = \frac{R}{Z^2} = s_y \cdot \mathrm{AM} \tag{3}$$

$$b = \frac{X}{Z^2} = s_y \cdot \mathrm{MP}$$

At frequencies above the resonant value f_o, X and b are positive and, according to the method here adopted, P lies above the horizontal axis. The frequency-scale runs counter-clockwise around the circle, having values zero and infinity at A.

In order to represent the *admittance of the entire resonating element* $RLCC_1$, we determine the point F, Fig. 2, such that $s_y \cdot a$ is numerically equal to ωC_1, where a denotes the distance AF. Since the damping constant of the resonator is assumed to be small, a may be considered as constant over the resonant range, which, as will appear, includes almost the entire circle.

The point F is to be taken as the origin of vectors for $RLCC_1$. FA is then negative, and FP is the vector sum of FA and AP. In other words,

$$s_y \cdot a = -\omega C_1$$

$$s_y \cdot \mathrm{AP} = Y = g + jb \tag{4}$$

$$s_y \cdot \mathrm{FP} = Y_1' = g_1' + jb_1'$$

Y_1' is the admittance of $RLCC_1$, having components $g_1' = g = s_y \cdot \mathrm{AM}$ and $b_1' = b - \omega C_1 = s_y \cdot \mathrm{WP}$. The phase angles for RLC and $RLCC_1$ are Θ and Θ_1 respectively. A positive phase angle (*e. g.* for point P in Fig. 2) indicates an inductive admittance and a lagging current. Over certain frequency ranges it is evident that Θ and Θ_1 have opposite signs. We may call the circle in Fig. 2 either the "admittance circle" or the "impedance circle." When it serves as an admittance circle, the frequencies run counter-clockwise, as indicated by the arrow Y in Fig. 2. On the impedance circle the direction is clockwise (arrow Z).

Impedance circle for the element $RLCC_1$. We perform a second inversion, this time taking F, Fig. 2, as center and $a = \mathrm{FA}$ as radius of the circle of inversion. The circle inverts into itself, and the point inverse to P is P'. This does not mean that FP' is the reciprocal of FP, for a differs in general from unity. It does mean, however, as is easily seen from geometrical considerations, that $\mathrm{FP}' = a^2/\mathrm{FP}$, whence if $s_y \cdot \mathrm{FP} = Y_1'$ we may write $Z_1' = 1/Y_1' = 1/s_y \cdot \mathrm{FP} = \mathrm{FP}'/s_y \cdot a^2 = s_z \cdot \mathrm{FP}'$, where s_z is the impedance scale-value, given by the equation

$$s_z = \frac{1}{a^2 s_y} = \frac{1}{2\rho R \omega^2 C_1^2} \text{ (ohms per div.)} \tag{5}$$

Conversely, if FP' represents the given admittance of $RLCC_1$, FP represents the impedance at the same frequency.

From Eqs. (4) and (5) it is evident that on the impedance diagram a represents the reactance of C_1.

Draw the horizontal axis Fx in Fig. 2, and the vertical line W'P'. We have the following relations for the impedance circle, analogous to Equations (4):

$$s_z \cdot a = -\frac{1}{\omega C_1} \tag{6}$$

$$s_z \cdot \mathrm{FP}' = Z_1' = \frac{1}{Y_1'} = R_1' + jX_1' \tag{7}$$

In Eq. (7) P' denotes any point on the impedance circle, the frequency being that characteristic of the inverse point P on the ad-

mittance circle. R_1' and X_1' are the equivalent series resistance and reactance of $RLCC_1$. Their values, which are easily derived, are now given:[3]

$$s_z \cdot \mathrm{FW}' = R_1' = \frac{R}{(1 - \omega C_1 X)^2 + \omega^2 C_1^2 R^2} \tag{8}$$

$$s_z \cdot \mathrm{W'P'} = X_1' = \frac{X - \omega C_1 (R^2 + X^2)}{(1 - \omega C_1 X)^2 + \omega^2 C_1^2 R^2} \tag{9}$$

It is sometimes convenient to regard the resonating element $RLCC_1$ as equivalent to a resistance R_1' and capacitance C_1' in series. The equivalent series capacitance, which for all points lying above the Fx axis assumes negative values, and which, like R_1', is a function of the frequency, is found from the equation $C_1' = -1/\omega X_1'$.

As is easily verified, the diameter of the $RLCC_1$ impedance circle, in ohms, is given by the maximum value of R_1':

$$s_z \cdot 2\rho = \frac{1}{\omega^2 C_1^2 R} \tag{10}$$

The frequency for maximum R_1' comes at point B on the impedance circle.

The element $RLCC_1$ may also be represented as a resistance R_1'' in parallel with a capacitance C_1''. These values are of course dependent upon the frequency. We find $R_1'' = 1/g_1' = 1/(s_y \cdot \mathrm{FW})$; $X_1'' = 1/b_1' = 1/(s_y \cdot \mathrm{WP}) = -1/\omega C_1''$. As is seen from Fig. 2, the minimum value of R_1'' occurs at point B on the admittance circle. C_1'' has maximum positive and negative values at the two quadrantal points P_1 and P_2 on the admittance circle, Fig. 2, that is, the points for which $\Theta = \pm 45°$.

It is interesting to note that at any frequency the impedance Z_1' of $RLCC_1$ is a mean proportional between the equivalent series and parallel resistances R_1' and R_1''. This is easily proved analytically or graphically.

3. *Calibration of the circle for frequency.* This process is most easily carried out for the admittance circle. To find the frequency corresponding to any point P′ on the impedance circle, one has only to find the inverse point P, Fig. 2, which locates the admittance corresponding to the given impedance, and then to find the frequency for P by the method about to be described.

[3] These values correspond to Eqs. (a) and (b) in Dye, l.c., p. 403.

The distribution of frequencies around the circle depends only on R, L and C and not at all on C_1. From Eqs. (1) and Fig. 2 we have

$$\tan \Theta = \frac{X}{R} = \frac{\omega^2 LC - 1}{\omega RC} = \frac{4\pi L}{R}(f - f_o) = \frac{2\pi n}{\alpha} = \frac{\text{BS}}{2\rho} = \frac{2\rho}{\text{AV}} \qquad (11)$$

approximately, since the resonance range is assumed small in comparison with the frequency. In (11), $\alpha = R/2L$ is the damping constant, and n is an abbreviation for the frequency-difference $f - f_o$. We may now define the frequency scale-value σ in cycles per division of the BS scale as follows:[4]

$$\sigma = \frac{n}{\text{BS}} = \frac{\alpha}{4\pi\rho} \qquad (12)$$

In order to find the frequency for point P we extend the line AP to meet the vertical axis through B at S. We then have for the frequency-difference $n = f - f_o$ between the points P and B,

$$n = \sigma \cdot \text{BS} \qquad (13)$$

For points below the diameter AB, BS is negative and $f < f_o$.

When P is to the left of the center of the circle, the distance BS becomes inconveniently great. In that case we use the distance AV, obtained by producing BP to meet the vertical axis through A at V. Then by Eq. (11) we have

$$n = \frac{\alpha\rho}{\pi} \cdot \frac{1}{\text{AV}} \qquad (14)$$

A method for the experimental determination of σ will be discussed in Sec. 12.

Since by Eq. (13) the frequency-scale on BS is linear, it is more convenient to use this equation whenever possible, rather than (14).

From (12) and (14) it is evident that the smaller the damping constant α the more wide-open is the frequency-scale (see Sec. 15).

4. *Critical Points on the Circular Locus.* Regarding Fig. 3 as the admittance diagram for $RLCC_1$, we have as before $\text{AF} = a = \omega C_1$ and $\text{AB} = 2\rho = 1/s_y R$. *At point B the frequency* f_o is that at which the reactance of RLC vanishes. For $RLCC_1$, with F as origin, B is the point at which, from Sec. 2, the equivalent parallel resistance R_1'' is a maximum. The frequency at B is still f_o. On the admittance

[4] Cf. Mallett, Telegraphy and Telephony, London 1929, p. 135.

circle the point P_7, inverse to B, gives the frequency for *maximum equivalent series resistance* R_1'; for upon inverting to an impedance diagram, P_7 inverts into B, and by Eq. (8) we then have $R_1' = s_z \cdot AB$. The frequency at B on the impedance circle is of course the same as at P_7 on the admittance circle.

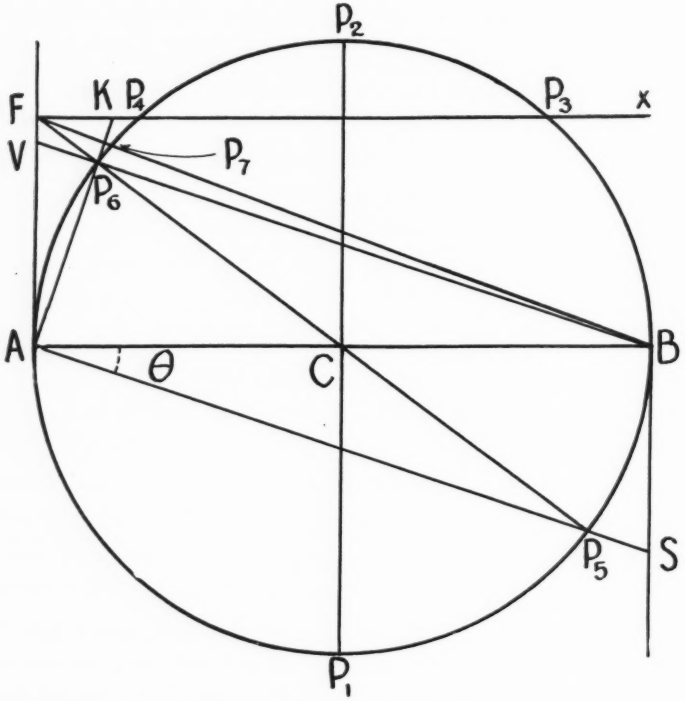

FIGURE 3. Critical Points on Circle Diagram for Resonating Element.

Consider next the mutually inverse points P_3 and P_4, at each of which the $RLCC_1$ resonator acts as a *pure resistance*. At P_3 the admittance, now a pure conductance, is large. This point has sometimes been called the point for "series resonance," since if C_1 is small,

P_3 is close to B, that is, the frequency is nearly that for series resonance of the *RLC* branch. On the other hand at P_4 we have the condition of "parallel resonance," in which C_1 plays a determining part. Upon inverting, we find that the high series resistance at parallel resonance is given by $s_z \cdot FP_3$. If a resonator is to show marked reduction in current at parallel resonance, C_1 and hence AF should be small. The distance AF is exaggerated in Fig. 3 beyond the value generally found for the piezo-electric resonator, in order to make the various points stand out more clearly. The values of R_1' and R_1'' become identical at P_3; the same is also true at P_4.

The quadrantal points are P_1 and P_2. On the admittance diagram, these are the points for maximum positive and negative values, respectively, of the equivalent parallel capacitance C_1'', as explained in Sec. 2. On the impedance circle, they are the points for greatest positive and negative equivalent series capacitance C_1'. Their frequencies are found from the points inverse to them, as explained at the beginning of Sec. 3.

The quadrantal points on the admittance circle are of greater usefulness than those on the impedance circle. The angular velocities at the admittance quadrantal points are well known to be

$$\omega_q = \sqrt{\omega_o^2 + \alpha^2} \pm \alpha \tag{15}$$

When α is small this may with sufficient accuracy be written $\omega_q = \omega_o \pm \alpha$. The following relations may also be mentioned, calling f_{q1} and f_{q2} the quadrantal frequencies on the admittance circle:

$$f_{q1} - f_{q2} = f_o^2 \tag{16}$$

$$\alpha = \pi(f_{q2} - f_{q1}) = \pi n_q = 2\pi(f_{q2} - f_o) = 2\pi(f_o - f_{q1}) \tag{17}$$

approximately, where n_q is the quadrantal frequency-range.

Many writers have found the points of *maximum and minimum impedance* of great importance. The relations between them, the corresponding frequencies, and the damping constant, are clearly brought to light by the present method. In Fig. 3, these points are P_5 and P_6. Using subscripts m and n to denote maximum and minimum values respectively, we have $Y_m = s_y \cdot FP_5$, $Y_n = s_y \cdot FP_6$, $Z_m = s_z \cdot FP_5$, $Z_n = s_z \cdot FP_6$. From the principle of inversion it is easily proved[5] that

[5] Kobzarew, Jour. Appl. Phys., Moscow, Vol. 6, p. 31, 1929.

$$Y_m Y_n = (\omega C_1)^2 \tag{18}$$

$$Y_m - Y_n = \frac{1}{R} \tag{19}$$

Thus when Y_m and Y_n are measured, their product gives C_1 and their difference gives R.

When R is small in comparison with $\sqrt{L/C}$, the frequency at maximum admittance is given[6] by

$$\omega^2 = \frac{C + C_1 + C_2}{LC\,(C_1 + C_2)} \tag{20}$$

5. *Constants of Resonating element in Terms of Y_m and Y_n.* We must first write the equations relating maximum and minimum admittance to the corresponding frequencies f_m and f_n. Referring to Fig. 3 and Eqs. (12) and (14), we find for the differences n_m and n_n between these frequencies and f_o,

$$f_o - f_m = n_m = \frac{\alpha}{4\pi\rho} \cdot \mathrm{BS} \quad \text{and} \quad f_n - f_o = n_n = \frac{\alpha\rho}{\pi} \cdot \frac{1}{\mathrm{AV}} \tag{21}$$

It is to be observed that BS = AV, and $\mathrm{BP}_6 = \mathrm{AP}_5$. From similar triangles and the above equations we then have

$$\frac{n_n}{n_m} = \frac{4\rho^2}{\mathrm{AV} \cdot \mathrm{BS}} = \frac{\mathrm{FP}_5}{\mathrm{FP}_6} = \frac{Y_m}{Y_n} \tag{22}$$

$$f_o - f_m = n_m = \frac{Y_n}{Y_m + Y_n}\,(f_n - f_m) \tag{23}$$

From (23) f_o may be precisely determined from observed values of f_m and f_n.

In terms of these same admittances and frequencies the damping constant may by Eq. (11) be expressed as follows (see Fig. 3):

$$\tan \Theta = \frac{\mathrm{BS}}{2\rho} = \frac{\mathrm{AV}}{2\rho} = \frac{\mathrm{FK}}{\mathrm{AF}} = \frac{\mathrm{FP}_6}{\mathrm{AF}} = \frac{Y_n}{\omega C_1} = \frac{2\pi n_m}{\alpha}$$

On substituting the value of n_m from (22) and of ωC_1 from (18), we have

[6] Boella, l'Elettrotecnica, Vol. 17, p. 672, 1930, or Proc. Inst. Radio Eng. Vol. 19, p. 1252, 1931.

$$\alpha = \frac{2\pi\sqrt{Y_m Y_n}}{Y_m + Y_n}(f_n - f_m) \tag{24}$$

In terms of frequency determinations alone, the damping constant may be derived from Eqs. (21), remembering that BS = AV:

$$\alpha^2 = 4\pi^2 n_m n_n \tag{25}$$

A comparison of Eq. (25) with (17) shows that the quadrantal frequency-range is a mean proportional between the frequency-differences n_m and n_n.

An expression for the self-inductance L can now be derived, by substituting from Eqs. (19) and (25) in the equation $\alpha = R/2L$:

$$L = \frac{1}{4\pi(Y_m - Y_n)\sqrt{n_m n_n}} = \frac{R}{4\pi\sqrt{Y_m Y_n}} \cdot \frac{Y_m + Y_n}{f_n - f_m} \tag{26}$$

When the frequency scale-value σ has been determined from observations, for example by the method outlined in Sec. 12 below, and when R and the diameter 2ρ are known, we may, by combining the equation for the damping constant, $\alpha = R/2L$, with Eq. (12), write the following very simple equation for L:

$$L = \frac{R}{8\pi\rho\sigma} \tag{26a}$$

From Eq. (17), L may be expressed as $L = R/n_q$. This equation is not likely to be as useful as (26) in the experimental determination of L. In general, observations of the two extreme values of admittance and of the corresponding frequencies are made more readily than for the quadrantal points, and as we have seen they are quite as useful in deriving the values of the electrical constants.

6. *Capacitance C_2 in Series with Resonating Element.* In the present section we will assume that the impedance Z_2 in Fig. 1 is a capacitance C_2. In the case of the piezo-electric resonator, C_2 may represent the air-gap between crystal and electrodes.

Fig. 4 represents the admittance diagram for $RLCC_1$ if we take as origin the point F such that $s_y a = \omega C_1$ where $a = \mathrm{AF}$. We shall prove that when C_2 is included in the circuit, the $RLCC_1C_2$ combination is equivalent to a *new resonating element* of the same form as

$RLCC_1$ having values $R'L'C'C_1'$ expressed in terms of R, L, C, C_1 and C_2, and independent of frequency.[7]

In Fig 4 let f_o be the frequency at B, and f that at any other point P. On inverting to an impedance diagram, the admittance FP becomes the impedance FP_1, of value $Z_1' = s_z \cdot FP_1$. This impedance is to be added vectorially to that of C_2. We lay off FF′ such that

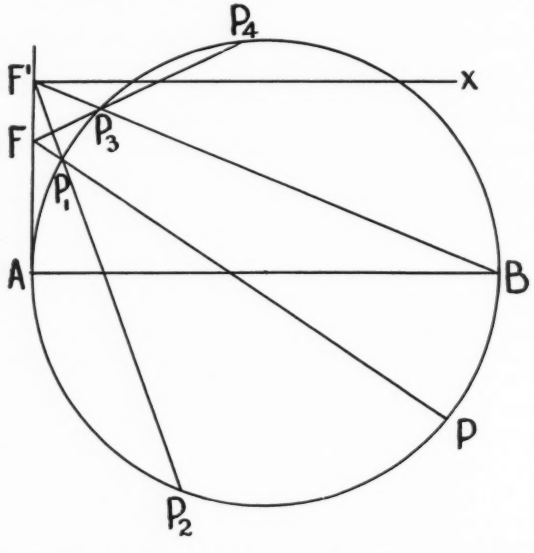

FIGURE 4. Graphical Construction for Capacitance C_2 in Series with Resonating Element.

$s_z \cdot F'F = -1/\omega C_2$. F′ becomes the new origin, the resistance axis being F′x, and $F'P_1$ is the vector sum of F′F and FP_1, so that for the impedance of $RLCC_1C_2$ we have $Z_2' = s_z \cdot F'P_1$. Let the distance AF′ be denoted by a'. From Eq. (6) we see that

$$-s_z a' = \frac{1}{\omega C_1} + \frac{1}{\omega C_2} \tag{27}$$

[7] See Watanabe, Proc. Inst. Radio Eng., Vol. 18, pp. 705–707, 1930.

Fig. 4 may be inverted back into an admittance diagram by performing inversions with F′ as center and a' as radius of the circle of inversion. Thus P_1 inverts into P_2, and the circle is the admittance locus for $RLCC_1C_2$ with F′ as origin of vectors. Besides the shift in origin, the introduction of C_2 has brought about two further changes: we have a new admittance scale-value s_y' given by

$$s_y' = \frac{1}{a'^2 s_z} \tag{28}$$

and the frequency-distribution around the circle is altered.

If the series reactance were inductive, the distance FF′ would of course be laid off in the opposite direction, and ωL_2 would replace $1/\omega C_2$.

The admittance of $RLCC_1C_2$, at the frequency corresponding to P on the Y circle for $RLCC_1$, is $Y_2' = s_y' \cdot \mathrm{F'P_2}$. This is the vector sum of two admittances, represented by F′A, which is practically constant over the resonance range, and $\mathrm{AP_2}$, which has a circular locus. Since this is exactly the condition that obtained in the case of the diagram for $RLCC_1$, it is clear that the new diagram represents a series chain, which we will call $R'L'C'$, in parallel with a fixed capacity C_1' which, from Eq. (27), has the value

$$C_1' = \frac{C_1 C_2}{C_1 + C_2} \tag{29}$$

The value of R' is found from Eqs. (2) and (28):

$$R' = \frac{1}{s_y' \cdot 2\rho} = R\left(1 + \frac{C_1}{C_2}\right)^2 \tag{30}$$

When the frequency-scale for the Y-circle for $RLCC_1C_2$ has been determined, L' and C' may be derived as in Sec. 5. The values are[8]:

$$L' = L\left(1 + \frac{C_1}{C_2}\right)^2 \tag{31}$$

$$C' = C\,\frac{C_2^2}{(C_1 + C_2)\,(C + C_1 + C_2)} \tag{32}$$

[8] Watanabe, l.c., p. 707. For a general treatment of the equivalent constants of networks, see T. E. Shea, Transmission Networks and Wave Filters, New York, 1929.

7. *Frequency-scale for $R'L'C'C_1'$.* This is easily derived from the frequency-scale for $RLCC_1$. The scale-value σ, as defined by Eq. (12), remains unaltered, since the damping constant $\alpha = R/2L = R'/2L'$ is unchanged. All frequencies are, however, shifted clockwise around the circle, so that with C_2 in, the resonant frequency f_o' at B is higher than the value f_o for $RLCC_1$ alone. The relation between f_o and f_o' is easily found graphically, by the following procedure.

In Fig. 4, $s_y' \cdot \mathrm{F'B} = Y_2'$ at frequency f_o'. On performing inversions we have $s_z \cdot \mathrm{F'P_3} = Z_2'$, $s_z \cdot \mathrm{FP_3} = Z_1'$, and $s_y \cdot \mathrm{FP_4} = Y_1'$. Hence $\mathrm{P_4}$ marks the point of frequency f_o' on the Y-circle for $RLCC_1$. On this latter circle the frequency at B is f_o; the frequency-difference between f_o and f_o' can be found graphically according to Sec. 3. Analytically, from (31) and (32), the difference is given by the equation

$$\omega_o' - \omega_o = \frac{1}{2\omega_o L(C_1 + C_2)} \tag{33}$$

8. *Capacitance C_3 in Parallel with Resonating Element.* The element may be either $RLCC_1$ or the equivalent $R'L'C'C_1'$ with C_2 in series. In either case the admittance locus is the circle in Fig. 2, with $s_y \cdot \mathrm{AF} = \omega C_1$ (or $\omega C_1'$). The addition of C_3 (which may be a condenser or the capacitance of lead wires) in parallel simply necessitates moving the origin upward exactly as in the case of a series capacitance C_2, Fig. 4, only here the new origin F' is determined by the equation $s_y \cdot \mathrm{FF'} = \omega C_3$, or $s_y \cdot a'' = \omega(C_1 + C_3)$. This does not alter the frequencies around the Y-circle, but all admittance values are of course changed, and the critical points $\mathrm{P_4}$, $\mathrm{P_5}$, $\mathrm{P_6}$ and $\mathrm{P_7}$, Fig. 3, come at lower frequencies and $\mathrm{P_3}$ at a higher frequency.

If it is desired to invert to an impedance diagram with C_3 in, the process is as before, except that a new scale-value s_z'' must be employed, given by the equation $s_y s_z'' = 1/a''^2$.

A similar construction holds for an inductance L_3 in parallel with $RLCC_1$.

9. *Introduction of Series Impedance Z_2 and Parallel Capacitance C_3.* Let the point P, Fig. 5, on the Y-circle for $RLCC_1$ be given, at any frequency f, and let it be required to find the vector representing the impedance of $RLCC_1Z_2$ at this frequency. The impedance of $RLCC_1$ is $Z_1' = s_z \cdot \mathrm{FP'}$. Draw the line OF of such length that $s_z \cdot \mathrm{OF} = Z_2 = R_2 + jX_2$, and with a slope making $s_z \cdot \mathrm{OM} = R_2$, $s_z \cdot \mathrm{MF} = X_2$. As represented in Fig. 5, X_2 is capacitive. If it were inductive, M would lie below F.

The vector sum of Z_2 and Z_1' is Z_2', given by the equation $Z_2' = s_z \cdot OP'$. Its components are $R_2' = s_z \cdot OW'$ and $X_2' = s_z \cdot W'P'$.

If now a circuit element, as C_3 Fig. 1, is to be placed in parallel with $RLCC_1Z_2$, it becomes necessary to invert to an admittance diagram. If O is taken as the center of inversion the circle remains the

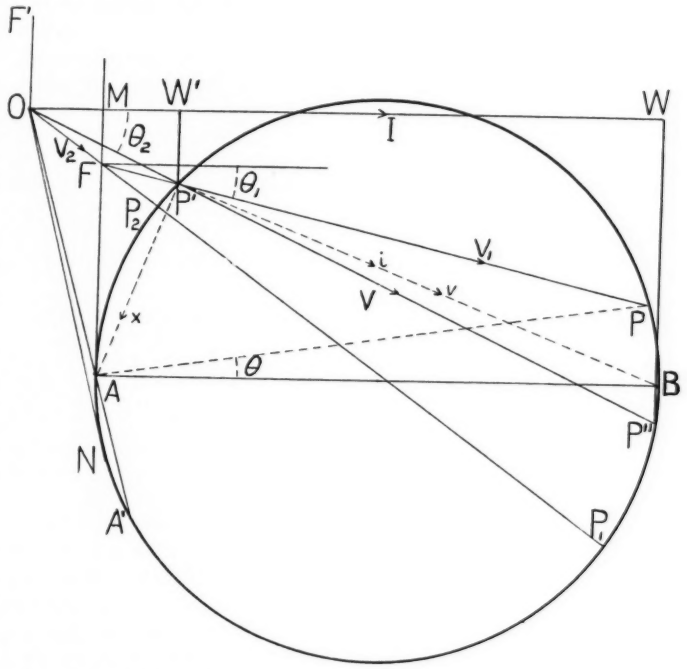

FIGURE 5. Graphical Construction for Impedance Z_2 in Series with Resonating Element.

same, and P′ inverts into P″. In performing this operation we must use as the radius of the circle of inversion the distance ON from O to the point of tangency on the circle. ON may be expressed in terms of ρ, C, C_1, and Z_2, and if we designate it by a_1'' we have for the new

scale-value on the admittance diagram for $RLCC_1Z_2$ $s_y'' = 1/(a_1''^2 s_y)$. The admittance of $RLCC_1Z_2$ at frequency f is then $Y_2' = s_y'' \cdot \mathrm{OP}''$.

As in Sec. 2, we find for the equivalent parallel resistance and capacitance of $RLCC_1C_2$, $R_2'' = 1/(s_y'' \cdot \mathrm{OW})$ and $X_2'' = -1/(\omega C_2'') = 1/(s_y'' \cdot \mathrm{WP}'')$.

Owing to the presence of R_2 in Z_2 it is not in general possible to replace $RLCC_1Z_2$ by an equivalent $R'L'C'C_1'$ in which the four parameters are independent of frequency.

With regard to the distribution of frequencies around the Y_2' circle, it is to be noted first that upon the addition of Z_2, A inverts into A′, which becomes the point for zero and infinite frequency. At points P_1 and P_2, on the extension of OF, frequencies are unchanged. All of the frequencies that on the $RLCC_1$ admittance circle lay on the lower (low frequency) portion between A and P are now compressed in the arc A′P″. All points on the admittance circle with Z_2 out (origin at F) are, upon the introduction of Z_2 (origin at O), shifted in both directions away from P_2 toward P_1. The frequencies are discussed further in Sec. 12.

The capacitance C_3 may now be taken into account by drawing a vertical reactance-axis through O and laying off a distance OF′ upward, such that $s_y'' \cdot \mathrm{OF}' = \omega C_3$. F′ is then the origin of admittance vectors for the network.

10. *Relative Phases of Current and Voltage Components.* Consider Fig. 5 as the impedance diagram for Z_2 in series with the resonating element. The arrows in this figure indicate relative phases, without regard to magnitudes of voltages and currents. Let the total current be denoted by I, its vectorial direction being horizontal as shown. Since by Sec. 9 the total impedance is $Z_2' = s_z \cdot \mathrm{OP}'$, it follows that the voltage V impressed upon the entire network lags behind I by the angle Θ_2. Similar reasoning shows that the voltage V_1 across the resonating element lags behind I by the the angle Θ_1, and that the voltage V_2 across Z_2 is parallel to OF. The direction of the current-vector for i, the current in the RLC branch, is obtained from P, the operating point on the admittance locus for $RLCC_1$. For we have by Equations (3) $Y = s_y \cdot \mathrm{AP}$, and since P is on the high-frequency side of the circle, i lags behind V_1 by the angle Θ. Now noting that the angles PP′B and PAB are equal, we see that PP′B also equals Θ, whence the direction of i relatively to I is given by the line P′B.

In the case of the piezo-electric resonator the velocity v of mechani-

cal vibration is in phase with i, while the mechanical strain x lags behind v by 90° and is represented by a vector parallel to P'A.

11. The case may arise in which it is desired to find the values of R, L, C, and C_1 from observations of the impedance of $RLCC_1Z_2$ at various frequencies. Z_2 may, for example, be the resistance of a thermo-element, and we will here assume it to be a pure resistance, R_2. There are several alternatives.

a. If only the maximum and minimum values Z_{2m}' and Z_{2n}' are observed, R and C_1 can be derived. We have

$$\frac{1}{R} = s_y'' \cdot 2\varrho = \frac{1}{Z_{2n}'} - \frac{1}{Z_{2m}'} \tag{34}$$

The value of C_1 is found from Eq. (18), in which Y_m and Y_n are the maximum and minimum admittances of $RLCC_1$. When R_2 is present, Y_m and Y_n must be derived from the observed Z_{2m}' and Z_{2n}'. This may be done graphically or analytically. For the graphical method, we construct an impedance diagram, Fig. 6. Starting with O as origin, the distance $OF = R_2/s_z$ is first laid off, and then $FD = (Z_{2m}' - Z_{2n}')/2s_z$, which is the radius of the impedance circle. To locate the center C, a perpendicular to FD is erected at D. The point C must then lie on this perpendicular, at such a distance from O that $s_z \cdot OC = (Z_{2m}' + Z_{2n}')/2$. The point A is found by drawing AC parallel to FD. C_1 may now be derived from the equation $FA = 1/(s_z \cdot \omega C_1)$. Upon drawing a line from O through C we have $s_z \cdot OP_1 = Z_{2n}'$, $s_z \cdot OP = Z_{2m}'$. The values of Z_m and Z_n for $RLCC_1$ are obtained by drawing a line, not shown in Fig. 6, from F through C. The distances from F to the points where this line cuts the circle, multiplied by s_z, give Z_m and Z_n. We may then compute C_1 according to Eq. (18): $(\omega C_1)^2 = Y_m Y_n = 1/Z_m Z_n$.

The algebraic expressions for Z_m and Z_n are readily deduced from geometrical considerations. For Z_n we find

$$Z_n = \frac{1}{2}\left\{\sqrt{(Z_{2m}' + Z_{2n}')^2 + 4R_2^2 - 4R_2(2R_2 + Z_{2m}' - Z_{2n}')} - (Z_{2m}' - Z_{2n}')\right\} \tag{35}$$

Z_m may be derived from the equation $Z_{2m}' - Z_{2n}' = Z_m - Z_n$.

b. If in addition to Z_{2m}' and Z_{2n}' we know also the corresponding frequencies f_{2m} and f_{2n}, we can determine L and C. In order to avoid

cumbersome expressions, it is usually advisable to derive the difference $f_n - f_m$ between the frequencies corresponding to Y_n and Y_m for $RLCC_1$, graphically according to Sec. 12. L is then found by Eq. (26),

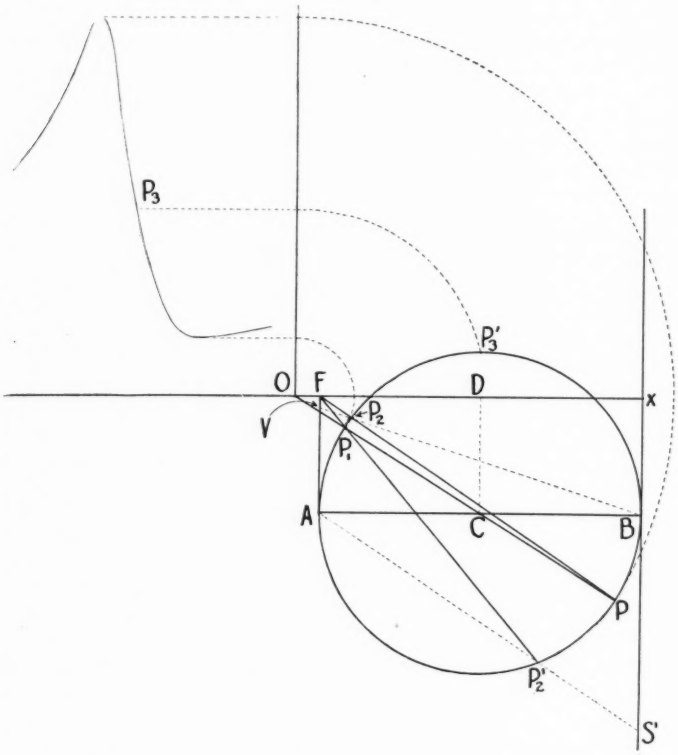

FIGURE 6. Resonance Curve with Associated Impedance Circle.

using values of Y_m and Y_n obtained graphically or derived from Eq. (35). From L and the resonant frequency, C is determined.

c. Greater precision is secured if impedances are observed at other frequencies in the neighborhood of resonance. R and C_1 must still be

found from the maximum and minimum values alone, but the other observations may be utilized for fixing the frequency-scale around the circle, as described in Sec. 12. If the observed impedances are plotted in the form of a resonance curve, with frequencies as abscissas, as shown in Fig. 6, then after the impedance circle has been drawn in the manner indicated above, points on the circle corresponding to various points on the curve may be located.[9] Thus in Fig. 6, P_3' on the circle is found from P_3 on the resonance curve. By the reverse process, the frequency f_o at point B may be found from the resonance curve. Also, by the use of the circular locus the phase-angle for any impedance may be found.

d. If in addition to the above mentioned data the phase angles corresponding to the various impedances are known, each observation may be used independently to fix one of the points on the impedance locus. The circle that best fits the points is then drawn. Each point thus established may be made to play its part in fixing the frequency scale, as described in the next section.

12. *Determination of Frequency Scale for the Circular Locus, from Observations with Z_2 in Series with the Resonating Element.* If only Z_{2m}', Z_{2n}', and the corresponding frequencies f_{2m} and f_{2n} are observed, we construct the impedance circle as in Sec. 11, and locate P and P_1 Fig. 6. By inverting these points with respect to F we obtain P_2 and P_2' such that $s_y \cdot FP_2 = Y_n$ and $s_y \cdot FP_2' = Y_m$ for $RLCC_1$. If P_2 and P_2' are not too close to A we draw lines AP_2 and AP_2', and produce them to meet the axis Bx at points S and S' (only S' is shown in Fig. 6). From Eq. (12) we then have as the frequency scale-value $\sigma = (f_{2m} - f_{2n})/SS'$ from which by interpolation the value of f_o at B can be found, as well as the frequency at any other point on the circle.

If one of the points, as P_2, is rather close to A, it is better, as was shown in Section 3, to draw the line BP_2V. From AV the corresponding value of BS is found from Eq. (11): $BS = 4\rho^2/AV$. The value of σ is then $(f_{2m} - f_{2n})SS'$ as indicated above.

When the locations and frequencies of a number of points on the circle are experimentally determined, as stated at the end of Section 11, we make a similar inversion for each point with respect to F, thus obtaining a number of values of AV or of BS, with accompanying frequencies. From the values of AV the corresponding BS values are

[9] Cf. Dye, l. c., p. 433, Mallett, l. c., p. 134; Mallett and Blumlein, Jour. Inst. El. Eng., Vol. 63, p. 397, 1925.

first computed by the equation $BS = 4\rho^2/\Delta V$, after which the average value of σ is found either graphically or algebraically. When σ has thus been determined, α may be found from Eq. (12) and L from the equation $\alpha = R/2L$.

If a line is drawn from F Fig. 6 through the center of the circle, it cuts the circle at two points which mark the maximum and minimum impedance for $RLCC_1$, as stated in Sec. 11a. By the method just described the two corresponding frequencies may be found.

When the method outlined above is followed, it is unnecessary to draw the resonance curve of Fig. 6.

13. *Effect of C_1, C_2, and Z_2 on Selectivity.* When anything beyond R, L, and C is taken into account, we can no longer use the damping constant $\alpha = R/2L$, for in all such cases the effective series R and L vary with frequency. Instead of this, we shall use as measure of selectivity the *change in phase accompanying a small change in frequency*, $\delta\Theta/\delta\omega$.[10]

As an illustration let us select two points close together, as P and Q, Fig. 7, on the admittance circle for $RLCC_1$. Then if the origin were at A (for RLC alone), the difference in phase-angle corresponding to (or causing) a frequency-difference PQ would be $\delta\Theta$. With C_1 in, the corresponding angular difference is $\delta\Theta_1$. If an impedance Z_2 is in series with $RLCC_1$ we first find P′ and Q′ inverse to PQ with respect to F; then using as origin the point O such that $s_z \cdot OF = Z_2$, as in Fig. 5, we draw OP′ and OQ′, giving the angular difference $\delta\Theta_2$. Thus, with Z_2 in, a very small change in phase will shift the frequency from P to Q.

If the series impedance is a capacitance C_2 instead of Z_2, the origin O will be on the extension of the line AF, as in Fig. 4, but the statements regarding selectivity remain otherwise unaltered.

For a given resonator and for a given operating frequency there is an optimum position of the point F, Fig. 7, such that the selectivity $\delta\Theta/\delta\omega$ is a maximum. In Fig. 7, F is shown approximately in the optimum position for an operating frequency at P. This position may be attained by assigning the proper value to the parallel capacitance C_1 or by the use of a series capacitance C_2 of suitable value, as indi-

[10] Mallett, l. c., p. 135, takes $\delta\omega/\delta \tan \Theta$ as measure of selectivity. It seems preferable to use the angle instead of its tangent for cases in which the origin is not at the point A, even though this makes the selectivity depend upon the frequency, because the change in phase accompanying a small change in frequency is an important criterion of resonator performance.

cated in Eq. (27). The insertion of a series resistance generally lowers the selectivity; nevertheless, if the distance AF is greater than the optimum value it may happen that a series resistance, by bringing the origin O slightly to the left of F, actually improves the selectivity somewhat. Still, such improvement can never be as great as that brought about by proper choice of C_1 or C_2.

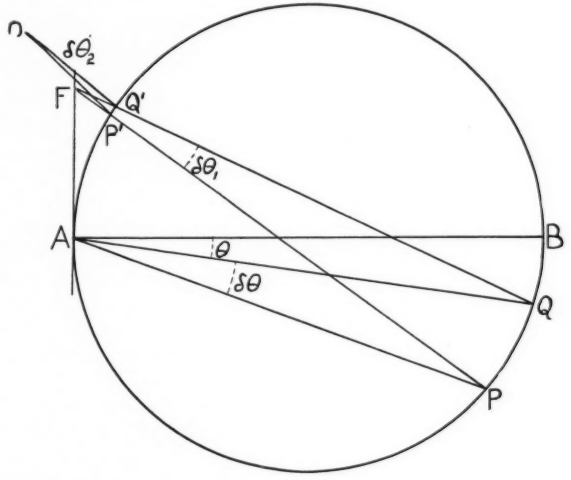

FIGURE 7. Effect of Additional Circuit Elements on Selectivity.

Whether the crystal is to be used as a resonator or an oscillator it is in general desirable to have the operating region in the neighborhood of the point B, where the frequency scale is wide open (See Sec. 15). For this region it is always an advantage in the interest of selectivity to have the origin near the point A.

A complete definition of selectivity should include the rate of change with respect to frequency of the *modulus* of the resonator impedance as well as that of its phase angle, since it is the rate of change in *vector* impedance that determines the selective properties of the resonator in any given network. The dependence of the modulus upon frequency may be obtained graphically from Fig. 7. For

frequencies in the neighborhood of P, and with A as origin (*RLC* branch alone), this variation may be expressed, for example, as

$$\frac{(AQ - AP)/AP}{\delta\omega}.$$

When O is the origin (Z_2 in series with $RLCC_1$), the expression is

$$\frac{(OQ' - OP')/OP'}{\delta\omega}.$$

This quantity, like $\delta\Theta/\delta\omega$, is in general greater the nearer the origin of vectors is to the point A.

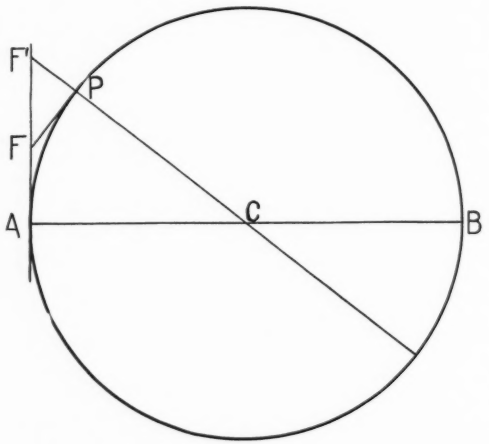

FIGURE 8. Graphical Construction for Determination of Condition for Maximum Voltage across an Air-gap.

14. *Condition for Maximum Voltage Across the Air-gap of a Piezo-electric Resonator.* As an illustration of the use of the impedance circle, let it be required to find the length of air-gap of a piezo-electric resonator necessary to build up maximum voltage across the gap with a given supply voltage. While the analytical solution is rather cumbersome, the solution is found graphically with great ease. The

problem was discussed by Dye[11] in relation to the luminous resonator of Giebe and Scheibe.

We seek the values of C_2 and f that make the ratio of X_2, the reactance of C_2, to Z_2' the total impedance of $RLCC_1C_2$, a maximum. In Fig. 8 is represented an impedance circle, the origin being at F for $RLCC_1$ and at F′ for $RLCC_1C_2$. We are dealing with a type of series resonance, in which the operating point P is on the high-frequency side of the circle. We have $s_z \cdot \mathrm{FA} = 1/\omega C_1$, $s_z \cdot \mathrm{F'F} = 1/\omega C_2 = X_2$, $s_z \cdot \mathrm{F'P} = Z_2'$. The problem is reduced to finding that position of F′ and also of P which makes $\mathrm{F'F/F'P} = \sin \mathrm{FPF'}/\sin \mathrm{F'FP}$ a maximum. It is not difficult to prove that this condition is satisfied when FP is tangent to the circle and the angle FPF′ is a right angle. The location of P determines the frequency, while the location of F′ determines C_2 and hence the length of air-gap. For a pronounced effect, C_1 should be as small as possible. It is to be noted that since F′P passes through the center C, the frequency is that of minimum impedance for the particular value of C_2, and that at this frequency the reactance of C_1 is numerically equal to the impedance of $RLCC_1$, because FA = FP. The point P is here unique in that it marks the same frequency on the impedance circle as on the admittance circle for $RLCC_1$.

15. *Frequency Calibration of the Admittance Circle as Affected by the Damping Constant.* From Eqs. (11) and (12), Sec. 3, it is evident that if the damping constant α decreases in a given ratio from α to α_1, the distance BS, Fig. 9, corresponding to any small frequency-difference $n = f_o - f$, where f_o is the frequency at B, must increase approximately in the same ratio, say from BS to $\mathrm{BS_1}$. Similarly, the distance SS′, corresponding to a small change δn in frequency from f to $f - \delta n$, becomes enlarged to S_1S_1' in the same ratio.[12] On the admittance circle for RLC, the arcs corresponding to δn are PP′ and $\mathrm{P_1P_1'}$ for large and small α respectively. We shall now prove that the arc corresponding to the difference between the same two frequencies f and $f - \delta n$, which obviously cannot increase in length everywhere around the circle in consequence of a diminution in α, increases until the quadrantal point $\mathrm{Q_1}$ is reached, and thereafter decreases toward zero. The same is true of the lower quadrantal point $\mathrm{Q_2}$ for frequencies below f_o. This property of the quadrantal points does not appear to have been mentioned hitherto, and it imposes a special interpreta-

[11] D. W. Dye, Proc. Phys. Soc. London, Vol. 38, p. 454, 1926.

[12] In Fig. 9, since P is on the high-frequency side of the circle, the values of n and δn are negative.

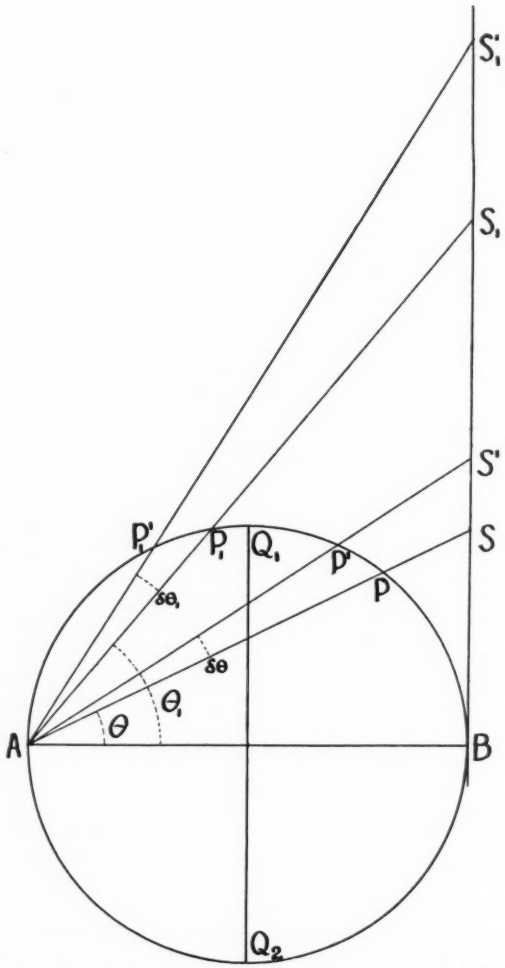

FIGURE. 9. Effect of a Change in Damping Constant upon Distribution of Frequencies around the Admittance Circle.

tion upon the statement that as the damping constant diminishes the frequency scale becomes "more wide-open."

It is convenient to use the small phase-angles $\delta\Theta$ and $\delta\Theta_1$ as the measures of the arcs, rather than the arcs themselves. Taking Θ as the phase-angle for damping constant α, it is easily shown from Eqs. (11) and (12) that

$$\frac{\delta\Theta}{\delta n} = \frac{2\pi}{\alpha}\cos^2\Theta = \frac{2\pi\alpha}{\alpha^2 + 4\pi^2 n^2} \tag{36}$$

It is evident from (36) that so long as n, and hence Θ, is small, $\delta\Theta/\delta n$ increases as α decreases, while for large values of n the opposite is true. An increase in $\delta\Theta/\delta n$ means an increase in the arc corresponding to δn, as α decreases.

We will call $\delta\Theta/\delta n = \psi$, and find the value of n for which ψ is independent of α. This value is found from Eq. (36) by setting the derivative of ψ with respect to α equal to zero. The result is the equation $\alpha = \pm 2\pi n$. Now it is well known (see Eq. (17) Sec. 4) that this relation between α and n is characteristic of the two quadrantal points. Hence the values of n for which $\delta\psi/\delta\alpha = 0$ are those at the quadrantal points.

The conclusion may be summed up in the statement that for one and the same frequency-interval any decrease in the damping constant increases the corresponding arc on the circle, and thereby the phase-interval $\delta\Theta$, so long as the arc is on the resonance side of the quadrantal points, and decreases it when the arc is on the other side.